

HADWIGER'S CONJECTURE FOR GRAPHS WITH INFINITE CHROMATIC NUMBER

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ABSTRACT. We construct a connected graph H such that

- (1) $\chi(H) = \omega$;
- (2) K_ω , the complete graph on ω points, is not a minor of H .

Therefore Hadwiger's conjecture does not hold for graphs with infinite coloring number.

1. NOTATION

In this note we are only concerned with simple undirected graphs $G = (V, E)$ where V is a set and $E \subseteq \mathcal{P}_2(V)$ where

$$\mathcal{P}_2(V) = \{\{x, y\} : x, y \in V \text{ and } x \neq y\}.$$

We also require that $V \cap E = \emptyset$ to avoid notational ambiguities. We denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. Moreover, for any cardinal α we denote the complete graph on α points by K_α .

For any graph G , disjoint subsets $S, T \subseteq V(G)$ are said to be *connected to each other* if there are $s \in S, t \in T$ with $\{s, t\} \in E(G)$. Note that K_α is a *minor* of a graph G if and only if there is a collection $\{S_\beta : \beta \in \alpha\}$ of nonempty, connected and pairwise disjoint subsets of $V(G)$ such that for all $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$ the sets S_β and S_γ are connected to each other. We will need the following observation later on:

Fact 1.1. *For any graph G , finite or infinite, the following are equivalent:*

- (1) G is connected;
- (2) if $S, T \subseteq V(G)$ are nonempty and disjoint such that $S \cup T = V(G)$ then S, T are connected to each other.

2. THE CONSTRUCTION

In [1], Hadwiger formulated his well-known and deep conjecture, linking the chromatic number $\chi(G)$ of a graph G with clique minors. His conjecture can be formulated that $K_{\chi(G)}$ is a minor of G for every graph G . In the following we present a connected graph H with chromatic number ω such that K_ω is not a minor of H . Let \mathbb{N} be the set of positive integers. For any $n \in \mathbb{N}$ we let

$$C_n = \{1, \dots, n\} \times \{n\}$$

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and set $V(H) = \bigcup_{n \in \mathbb{N}} C_n$. As for the edge set of H , we define

$$E(H) = \{ \{(1, n), (1, n+1)\} : n \in \mathbb{N} \} \cup \bigcup_{n \in \mathbb{N}} \mathcal{P}_2(C_n).$$

Proposition 2.1. $\chi(H) = \omega$.

Proof. Since we have $\text{card}(V(H)) = \omega$ we get $\chi(H) \leq \omega$. Moreover, each C_n is a complete subgraph of H , so H cannot be colored with finitely many colors. \square

For the remainder of this note, we assume that $\{S_n : n \in \omega\}$ is a collection of nonempty, connected, pairwise disjoint subsets of H such that for $m \neq n$ the sets S_n, S_m are connected to each other. Our goal is to show that such a collection cannot exist.

First, we need a simple observation on what a connected subset of H looks like. If $S \subseteq V(H)$ we define $I(S) = \{n \in \mathbb{N} : C_n \cap S \neq \emptyset\}$.

Lemma 2.2. *Suppose $S \subseteq V(H)$ is connected and $m < n \in I(S)$. Then for all $x \in \mathbb{N}$ with $m \leq x \leq n$ we have $(1, x) \in S$.*

Proof. If $(1, m) \notin S$ then $T = S \cap C_m$ and $S \setminus T$ are disjoint, nonempty and not connected to each other. By Fact 1.1, S is not connected, contradicting our assumption. A similar argument shows that $(1, n) \in S$. Suppose there is x with $m < x < n$ and $(1, x) \notin S$. Then set $T = \{(i, j) \in S : j < x\}$. Again, T and $S \setminus T$ are nonempty and not connected to each other, so S is not connected, contradicting our assumption. \square

If $\{S_n : n \in \omega\}$ is a collection of subsets of $V(H)$ as described above, then for every $k \in \mathbb{N}$ the set of neighbors of S_k , which is denoted by $N(S_k)$, must be infinite. As the next lemma shows, this implies that $I(S_k)$ must be infinite for all $k \in \mathbb{N}$.

Lemma 2.3. *If $S \subseteq V(H)$ is such that $I(S)$ is finite, then $N(S)$ is finite.*

Proof. Let $m = \max(I_S)$. Then $N(S) \subseteq \bigcup_{i=1}^{m+1} C_i$, which is a finite set. \square

Now we go back to our assumption that $\{S_n : n \in \omega\}$ is a collection of nonempty, connected, pairwise disjoint subsets of H such that for $m \neq n$ the sets S_n, S_m are connected to each other. We consider just two of these sets, say S_0, S_1 . Because of lemma 2.3, the sets $I(S_0)$ and $I(S_1)$ are infinite. For $k = 0, 1$ let $\mu_k = \min(I(S_k))$. We may assume that $\mu_0 \leq \mu_1$. Since $I(S_0)$ is infinite, there is $n \in I(S_0)$ with $n \geq \mu_1$. So lemma 2.2 implies that $(1, \mu_1) \in S_0 \cap S_1$, contradicting the assumption that the S_k are pairwise disjoint. So we established:

Proposition 2.4. *The complete graph K_ω is not a minor of H .*

REFERENCES

- [1] Hadwiger, Hugo, *Über eine Klassifikation der Streckenkomplexe*, Vierteljschr. Naturforsch. Ges. Zürich, **88** (1943), 133–143.

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